YEAR 12 MATHEMATICS
ASSESSMENT TASK TERM 2 2005

QUESTION 1: START A NEW PAGE

Marks

(a) The roots of the equation $2x^2 + 3x = 4$ are α and β.
Find the value of:
(i) $\alpha + \beta$ **1**
(ii) $\alpha^2 + \beta^2$ **2**

(b) A water tank which was full, developed a leak, such that the volume of water V Litres in a tank is decreasing according to the equation:

$$\frac{dV}{dt} + kV = 0, \text{ where } k \text{ is a positive constant.}$$

(i) Show that $V = Ae^{-kt}$, where A is a constant, is a solution to the above equation. **1**
(ii) If the tank is half full after 7 days of leaking, show that the decay rate constant (k) is equal to $ln2$ per week. **1**
(iii) How much water, to the nearest litre, has been lost 4 weeks from the time the tank began to leak, if the full tank held 5000 L? **2**

(c) In archery John usually hits the centre of a target with 3 out of 4 attempts, while Susan hits it with 1 out of 3 attempts.
(i) If John and Susan shoot one arrow each, what is the probability that:
(α) Both arrows hit the centre of the target? **1**
(β) At least one arrow hits the centre of the target? **1**

(ii) In a particular archery competition, the winner is the first person to hit the centre of the target. If they take turns to shoot and Susan shoots first, what is the probability that Susan wins with her second arrow? **2**

(d) A particle, starting from rest, moves along the x-axis so that its velocity v cm/s, after t seconds, is given by:

$$v = -3\cos(2\pi\, t).$$

(i) Find the exact velocity of the particle at $0 \cdot 5$ seconds. **1**
(ii) Draw a velocity-time graph for the particle for $0 \le t \le 2$. **2**
(iii) How many times does the particle change direction in the first 2 seconds? **1**

QUESTION 2: START A NEW PAGE

(a) A particular rare plant has a 20% chance of producing fruit if it is grown from seed.
(i) If 3 seeds are sown, what is the probability, as a percentage, that only 2 of the plants will produce fruit? **2**
(ii) What is the minimum number of seeds which need to be sown to ensure that there is a $99 \cdot 9$ % chance that at least one plant will produce fruit? **3**

Marks

(b) (i) Show, for the expression in x: **2**

$(x-l)(x-m)-9n^2$, where l, m and n are real numbers,

that the discriminant, Δ, is given by:

$$\Delta = l^2 - 2lm + m^2 + 36n^2 .$$

(ii) Hence, or otherwise, show that $(x-l)(x-m)-9n^2 = 0$ has real roots. **2**

(c) The acceleration, $\ddot{x}$ ms^{-2}, of a particle travelling in a straight line at time t seconds, is given by:

$$\ddot{x} = \frac{5}{(t+1)^2} .$$

The particle begins its motion at the origin, with a velocity of 5 ms^{-1}.

(i) Show that the velocity, $\dot{x}$ ms^{-1}, of the particle at time t seconds is given by: **2**

$$\dot{x} = 10 - \frac{5}{t+1} .$$

(ii) Explain why the particle is never stationary. **1**

(iii) Find an expression for the displacement of the particle at time t seconds. **2**

(iv) Briefly describe the motion of the particle as t increases. **1**

QUESTION 3: START A NEW PAGE

(a) The populations of two towns A and B were 10 000 and 15 000 on 1 January 2000. The populations of the towns are growing exponentially according to the formulae $P_A = 10\,000e^{0\cdot 2t}$ and $P_B = 15\,000e^{0\cdot 15t}$ respectively, where t is the number of years since 1 January 2000.

(i) During which year will the towns have the same population? **2**

(ii) At what rate is the population of town A increasing at this time? **2**

(b) Two dice are thrown at the same time. One is a standard die which has the sides labelled with the numbers 1, 2, 3, 4, 5, 6, while the other has its sides numbered 3, 4, 5, 6, 6, 6. The final score is the sum of the two uppermost faces.

(i) Draw a diagram indicating all the possible outcomes in the sample space. **1**

(ii) Find the probability when the final score is:

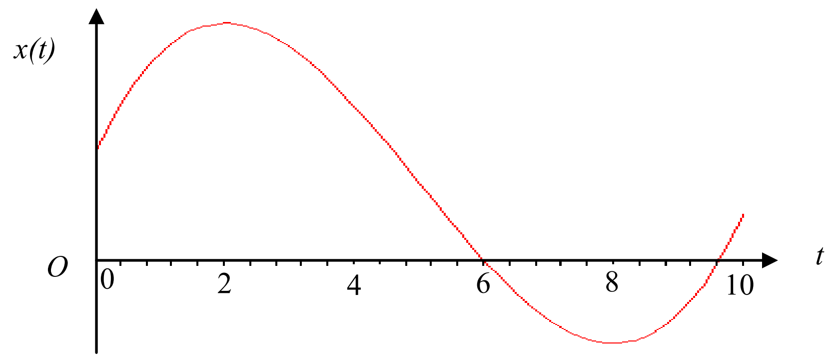
(α) 11? **1**

(β) Less than 9? **1**

(iii) If one die is chosen at random and rolled twice what is the probability that the final score is greater than 10? **2**

Marks

(c) A particle P moves along a straight line for 10 seconds, starting at the fixed point A, at time $t = 0$. At time t seconds, P is $x(t)$ metres to the right of the origin O. The graph of $x(t)$ is shown in the diagram below.



(i) Sketch the velocity-time graph for the particle. **2**

(ii) Estimate the time(s) when the particle:

(α) Is at rest. **1**

(β) Has zero acceleration. **1**

(γ) Is moving in the negative direction. **1**

(δ) Is to the left of A. **1**

Question 4: START A NEW PAGE

(a) Solve for x: $2(x-3)(x-\pi) > 0$. **2**

(b) The number of bacteria N, at time t seconds, in a forensic sample can be described by the formula $N = N_0 e^{kt}$, where k and N_0 are constants. When the sample was first tested at 3pm it contained 500 000 bacteria. Two hours later there were 700 000 bacteria.

(i) Show that the rate of change of N is proportional to N. **1**

(ii) At what time, to the nearest minute, was the number of bacteria 250 000? **3**

(c) A particle moves on the x-axis so that its displacement, x m, from the origin O, at time t seconds, is given by:

$$x = 2\cos t + t\sin t.$$

(i) Write an expression for $\dot{x}$. **2**

(ii) Show that the particle is at rest when $t = \tan t$. **1**

(iii) Show that when the particle is at rest its displacement is: **2**

$x = (T^2 + 2)\cos T$, where T is a solution to $t = \tan t$.

(iv) Sketch, on the same axes, the graphs of: **2**

$$y = \tan t \text{ and } y = t, \text{ for } 0 \le t \le \frac{3\pi}{2}.$$

(v) Hence, or otherwise, show that the first two positions of rest are on opposite sides of the origin. **2**

END OF EXAMINATION

YEAR 12 MATHEMATICS ASSESSMENT TEST TERM 2
2005

1(a) $2x^2 + 3x - 4 = 0$

(i) $\alpha + \beta = -b/a = -3/2$ ①

(ii) $\alpha\beta = c/a = -2$ ①

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \frac{9}{4} + 4$

$= 6\frac{1}{4}$ ①

1(b) (i) $V = Ae^{-kt}$

$\frac{dV}{dt} = -kAe^{-kt}$

LHS $= \frac{dV}{dt} + kV = -kAe^{-kt} + kAe^{-kt}$ ①

$= 0$

$=$ RHS

$\therefore V = Ae^{-kt}$ is a solution

(ii) $t = 0$ $V = A$

$t = 1$ $V = \frac{1}{2}A$

$\therefore \frac{1}{2}A = Ae^{-k \times 1}$

$\therefore \frac{1}{2} = e^{-k}$

$\therefore \ln\frac{1}{2} = -k$ ①

$\therefore k = \ln 2$ /week

(iii) $t = 4$ $V = 5000e^{-4k}$

$= 5000e^{-4\ln 2}$

$= 312.5$

$\therefore$ Vol lost $= 5000 - 312.5$

$= 4687.5L$

$= 4688L$ (nearest Litre)

1(c) (i) (α) $P(J) = \frac{3}{4}$ $P(S) = \frac{1}{3}$

$P(\text{Both}) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ ①

(β) $P(\text{at least 1 hit}) = 1 - P(\text{Both miss})$

$= 1 - \frac{1}{4} \times \frac{2}{3}$

$= 1 - \frac{1}{6} = 5/6$ ①

(ii) $P(S \text{ wins}) = P(S \text{ miss}) \times P(J \text{ miss}) \times P(S \text{ HITS})$

$= \frac{2}{3} \times \frac{1}{4} \times \frac{1}{3}$ ($1\frac{1}{2}$)

$= \frac{1}{18}$ ($\frac{1}{2}$)

(some working or explanation must be given for 2 marks)

1(d) $v = -3\cos(2\pi t)$

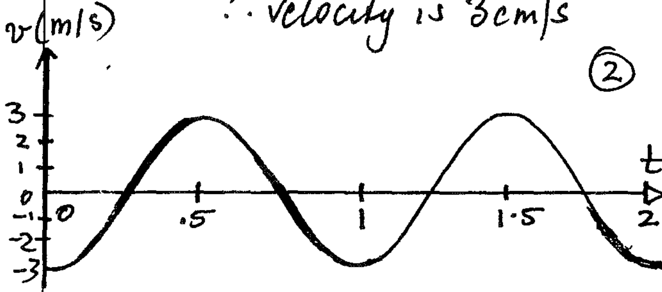
(i) $t = 0.5$ $v = -3\cos(2\pi \times .5)$

$= -3\cos\pi$

$= 3$ ①

$\therefore$ velocity is 3 cm/s

②



(iii) direction changes when sign of v changes sign

$\therefore$ 4 times ①

(2) (a)(i) $P(2) = 3 \times (\cdot 2)(\cdot 2)(\cdot 8)$ ① ①

$= 0.096 = 9.6\%$

(ii) $P(\text{at least one}) = 1 - P(\text{none})$

$\therefore \frac{99.9}{100} > 1 - (\cdot 8)^n$ ①

$\therefore (0.8)^n < 1 - 0.999$

$\therefore (0.8)^n < 1 \times 10^{-3}$

$\therefore n > \frac{\log(1 \times 10^{-3})}{\log(0.8)}$ ①

$n > 30.95655348$ ①

$\therefore$ plant 31 seeds

(b) (i) $(x-l)(x-m) - 9n^2$

$= x^2 - (l+m)x + lm - 9n^2$

$\Delta = b^2 - 4ac$

$\Delta = (l+m)^2 - 4(1)(lm - 9n^2)$

$\Delta = l^2 + 2lm + m^2 - 4lm + 36n^2$

$\Delta = l^2 - 2lm + m^2 + 36n^2$

(ii) $\Delta = (l-m)^2 + 36n^2$ ①

both terms are perfect squares

$\therefore \Delta \geq 0$ ①

$\therefore$ equation has real roots

2 (c) (i) $\ddot{x} = \frac{5}{(t+1)^2}$

$t=0 \quad \dot{x} = 5$

$\dot{x} = \int \frac{5}{(t+1)^2}\, dt$

$= \frac{-5}{(t+1)} + c$ ①

$t=0 \quad \dot{x}=5 \quad c=10$

$\therefore \dot{x} = \frac{-5}{(t+1)} + 10$ ①

[Alternative solution
show $\dot{x}=5$ when $t=0$ ①
and show $\frac{d\dot{x}}{dt} = \ddot{x}$] ①

(ii) for stationary $\dot{x}=0$

$\therefore 0 = \frac{-5}{t+1} + 10$

$\therefore t = -\frac{1}{2}$ ①

but $t \geq 0 \quad \therefore \dot{x} \neq 0$

$\therefore$ never stationary.

[Alternative solution
$\dot{x} = \frac{-5}{t+1} + 10$
For $t \geq 0 \quad \frac{-5}{t+1} > -10$
$\therefore \dot{x} > 0 \quad \therefore$ never stationary] ①

(iii) $\dot{x} = \frac{-5}{t+1} + 10$

$x = \int \left(\frac{-5}{t+1} + 10\right) dt$ ①

$\therefore x = -5\ln(t+1) + 10t + k$

$t=0 \quad x=0 \quad \therefore k=0$

$x = 10t - 5\ln(t+1)$ ①

(iv) The particle moves in a positive direction from the origin, at 5m/s and continues to accelerate to a velocity ~~of~~ approaching 10m/s as time increases.

(minimum answer: particle accelerates in positive direction fro[m] origin.) ①

3(a) (i) $10000e^{0.2t} = 15000e^{0.15t}$ ($\frac{1}{2}$)

$\therefore \frac{10}{15} = \frac{e^{0.15t}}{e^{0.2t}}$

$\therefore \frac{10}{15} = e^{-0.05t}$ ($\frac{1}{2}$)

$t = \frac{\ln\left(\frac{10}{15}\right)}{-0.05t}$

$t = 8.109302162$ ($\frac{1}{2}$)

$\therefore$ equal population after 8.1 years from 1 Jan 200[0]

$\therefore$ Populations equal durin[g] 2008 ($\frac{1}{2}$)

(ii) $\frac{dP_A}{dt} = 0.2 \times 10000e^{0.2t}$ ①

$t = 8.109302162\cdots$

$= 10125$ people/year [①]

(Accept ± 1 person)

(b) (i)

Die ① (across), Die ② (down)

	1	2	3	4	5	6
1	4	5	6	7	8	9
2	5	6	7	8	9	10
3	6	7	8	9	10	11
6	7	8	9	10	11	12
6	7	8	9	10	11	12
6	7	8	9	10	11	12

①

(ii) (α) $P(11) = \frac{4}{36} = \frac{1}{9}$

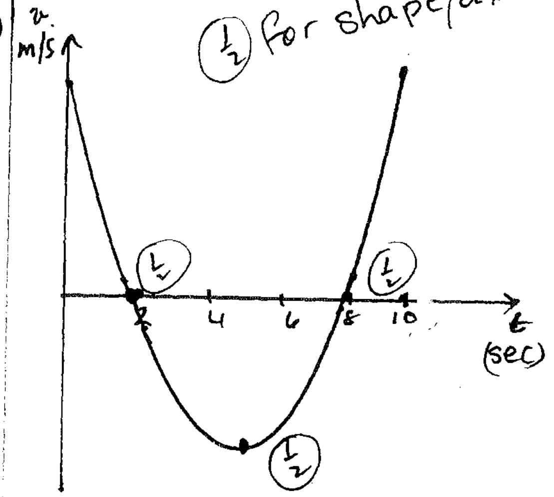
(β) $P(\text{Less than } 9) = \frac{18}{36} = \frac{1}{2}$

(iii) $P(>10)$ Normal Die $= \frac{3}{36}$ ①

$P(>10)$ Other Die $\frac{15}{36}$

$P(>10) = \frac{1}{2} \times \frac{3}{36} + \frac{1}{2} \times \frac{15}{36} = \frac{1}{4}$

[some working or explanation must be given for full marks]

3(c) 

(ii) (α) Rest $\dot{x} = 0$

$t = 2$ (1/2) and $t = 8$ sec (1/2)

(β) $\ddot{x} = 0$ $t = 5$ sec (1)

(γ) $\dot{x} < 0$ $2 < t < 8$ sec.

(must have < signs)

(δ) $t > 4 \cdot 6$ sec (1)

(Accept $\pm 0 \cdot 2$ for all answers)

(4) (a) $2(x-3)(x-\pi) > 0$

for $2(x-3)(x-\pi) = 0$

$x = 3$ or $x = \pi$ (1)

$\therefore x < 3$ or $x > \pi$ (1)

(b) (i) $N = N_0 e^{kt}$

$\frac{dN}{dt} = k(N_0 e^{kt})$

$= kN$ (1)

$\therefore \frac{dN}{dt} \propto N$

(ii) $700000 = 500000\, e^{2k}$

$2k = \ln\left(\frac{7}{5}\right)$

$\therefore k = \frac{1}{2}\ln\left(\frac{7}{5}\right)$ (1)

$250000 = 500000\, e^{\frac{1}{2}\ln(\frac{7}{5})t}$

$\therefore \frac{1}{2} = e^{\frac{1}{2}\ln(7/5)t}$

$\therefore t = \frac{\ln\frac{1}{2}}{\frac{1}{2}\ln\left(\frac{7}{5}\right)}$ (1)

$\therefore t = -4 \cdot 12$ hours (1/2)

= 4 hours 7 mins before 3pm

$\therefore$ time was 10:53 am (1/2)

(c) (i) $x = 2\cos t + t\sin t$ (1)

$\dot{x} = -2\sin t + t\cos t + \sin t$

$\dot{x} = -\sin t + t\cos t$ (1)

$= t\cos t - \sin t$

(ii) Rest $\dot{x} = 0$

$\sin t = t\cos t$

$\therefore t = \frac{\sin t}{\cos t}$

$\therefore t = \tan t$ (1)

(iii) $x = 2\cos t + t\sin t$ (1)

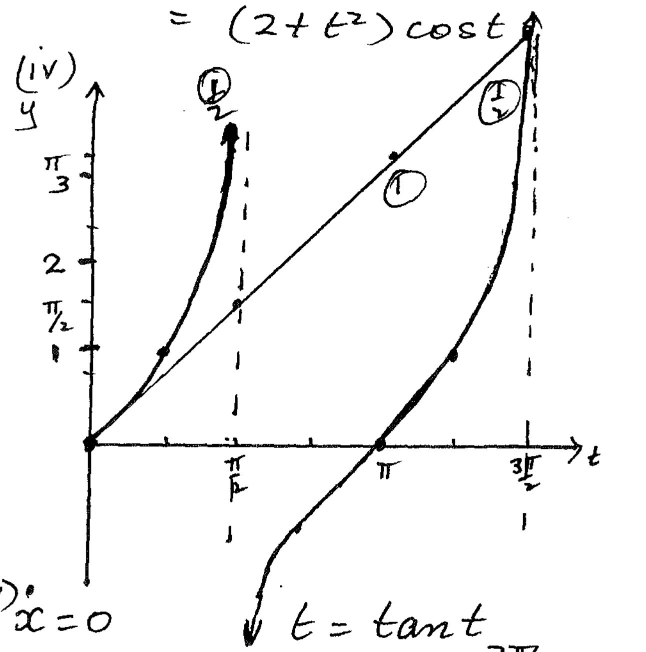
$= 2\cos t + \tan t \sin t$

$= 2\cos t + \tan^2 t \cos t$

$= \cos t\,[2 + \tan^2 t]$

$= \cos t\,[2 + t^2]$ (1)

$= (2+t^2)\cos t$

(iv) 

(v) $\dot{x} = 0$ $t = \tan t$

$\therefore t = 0$ or $\pi/2 < t < 3\pi/2$

(1) $\therefore x = 2$ or $x = (2+t^2)\cos t$

but $\cos t < 0$ for $\pi/2 < t < 3\pi/2$

$\therefore (2+t^2)\cos t < 0$ (1)

for 2nd solution of t.

$\therefore$ particle is at rest on opposite sides of the origin